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MATHEMATICS

A MEAN-VALUE THEOREM FOR $\zeta(s, w)$

BY

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In the theory of the RIEMANN zeta-function $\zeta(s)$ $(s = \sigma + it)$, the problem of the order of magnitude of this function in the critical strip $0 \leqq \sigma \leqq 1$ plays an important rôle. Though there are many contributions to the subject, the problem seems still far remote from its solution [1]). In many investigations instead of $\zeta(s)$ one considers the more general function $\zeta(s, w)$, which involves a real parameter w satisfying

$$(1) \qquad 0 < w \leqq 1,$$

and which originates from the series

$$(2) \qquad \zeta(s, w) = \sum_{n=0}^{\infty} \frac{1}{(n+w)^s} = \sum_{n=0}^{\infty} e^{-s \log (n+w)} \quad (\log (n + w) \text{ real}, \ \sigma > 1).$$

To a great extent the results obtained for $\zeta(s)$ remain valid also for $\zeta(s, w)$, the argument not being complicated too much by the introduction of the parameter w. For sake of convenience we write

$$\zeta^*(s, w) = \zeta(s, w) - \frac{1}{w^s},$$

which has some advantage, as $\zeta^*(s, w)$ also is defined for $w = 0$ (cf. (2)). Now generally spoken the results for $\frac{1}{2} \leqq \sigma \leqq 1$ and $\sigma = 1$ respectively take the form

$$(3) \qquad \zeta^*(\sigma + it, w) = O(t^{\lambda(\sigma)}) \ (t > 0),$$

where $\lambda(\sigma)$ is some positive function only depending on σ, and

$$(4) \qquad \zeta^*(1 + it, w) = O\left(\frac{\log t}{\log \log t}\right) \qquad (t > 3).$$

Much more however is known about the average order, e.g.

$$\frac{1}{T} \int_1^T |\zeta(\sigma + it)|^2 dt \backsim \zeta(2\sigma) \ (\sigma > \tfrac{1}{2}).$$

The aim of this paper is to investigate the following mean value

$$(5) \qquad \int_0^1 |\zeta^*(s, w)|^2 dw,$$

which as far as we are aware till now is not dealt with. Although with respect to the order of magnitude of $\zeta^*(s, w)$ only estimates like (3) and (4) are known, it turns out that the expression (5) can be estimated much sharper. In fact we shall prove the following theorems.

[1]) Cf. E. C. TITCHMARSH, The theory of the Riemann zeta-function, (Oxford, 1951), especially Chapters IV and VII.

Theorem 1. *There exists a positive constant A_0, such that if A is any constant $\geqq A_0$, and if $s = \sigma + it$ (σ, t real) is restricted to the region*

$$|t| \geqq 3, \frac{1}{2} + \frac{1}{2A \log |t|} \leqq \sigma \leqq 1,$$

we have

$$\int_0^1 |\zeta^* (s, w)|^2 \, dw = \frac{1}{2\sigma - 1} + \Theta \frac{2A \log |t|}{|t|^{2\sigma-1}},$$

where $|\Theta| \leqq 1$. For A_0 for instance the choice $A_0 = 32$ is permitted.

In a less detailed form theorem 1 obviously may be stated as follows.

Theorem 1*. *If σ_0 is a constant $> \frac{1}{2}$ and < 1, then we have*

$$\int_0^1 |\zeta^* (s, w)|^2 \, dw = \frac{1}{2\sigma - 1} + O\left(|t|^{-(2\sigma-1)} \log |t|\right)$$

uniformly in $\sigma_0 \leqq \sigma \leqq 1$.

Theorem 2. *If t is real and $|t| \geqq 3$, then we have*

$$\int_0^1 |\zeta^*(\tfrac{1}{2} + it, w)|^2 dw < B \log |t|,$$

where for instance we may take $B = 34$.

The proofs of these theorems are preceded by five lemma's; lemma 1 and lemma 2 form a straightforward generalization of wellknown analogous results for $\zeta(s)$ [2]).

Preliminary remarks

If $z = x + iy$ (x, y real) is no real integer, we have

$$i + \cot \pi z = - \frac{2i\, e^{2\pi i z}}{1 - e^{2\pi i z}}, \quad -i + \cot \pi z = \frac{2ie^{-2\pi i z}}{1 - e^{-2\pi i z}}.$$

Hence

$$|i + \cot \pi z| \leqq \frac{2e^{-2\pi y}}{1 - e^{-2\pi y}}, \text{ if } y > 0, \tag{6}$$

and

$$|-i + \cot \pi z| \leqq \frac{2e^{2\pi y}}{1 - e^{2\pi y}}, \text{ if } y < 0. \tag{7}$$

If moreover x is half an odd integer, we have

$$|i + \cot \pi z| = \frac{2e^{-2\pi y}}{1 + e^{-2\pi y}} \leqq 1, \text{ if } y \geqq 0 \tag{8}$$

$$|-i + \cot \pi z| = \frac{2e^{2\pi y}}{1 + e^{2\pi y}} \leqq 1, \text{ if } y \leqq 0. \tag{9}$$

If X is a non-integral positive number and if p denotes an integer $> X$, let K_p denote the broken line with successive vertices

$$X - i\infty,\ X - ip,\ p + \tfrac{1}{2} - ip,\ p + \tfrac{1}{2} + ip,\ X + ip,\ X + i\infty.$$

Let S denote the set of points z, belonging either to the line $Re\, z = X$

2) Cf. E. C. Titchmarsh, l.c., § 4.14.

or to one of the broken lines K_p (p integral and $> X$). Then there exists a constant $K = K(X)$, such that

$$|\cot \pi z| \leqq K, \text{ if } z \in S. \tag{10}$$

Lemma 1. *If X is a non-integral positive number and if $\sigma > 1$, $0 \leqq w \leqq 1$, then we have*

$$\sum_{n>X} \frac{1}{(n+w)^s} = -\frac{1}{2i} \int_{X-i\infty}^{X+i\infty} (z+w)^{-s} \cot \pi z \, dz, \tag{11}$$

where the integral is taken along the straight line $Re\, z = X$, and where $(z+w)^{-s}$ means the principal value [3]).

Proof. If $z = x+iy$ (x, y real) and if $x > 0$, then we have $Re(z+w) > 0$, on account of $0 \leqq w \leqq 1$, hence

$$\left\{ \begin{aligned} &|(z+w)^{-s}| = |e^{-(\sigma+it)(\log|z+w| + i \arg(z+w))}| \\ &\leqq |z+w|^{-\sigma} e^{\frac{1}{2}\pi|t|} \leqq \min(x^{-\sigma}, |y|^{-\sigma}) \cdot e^{\frac{1}{2}\pi|t|}. \end{aligned} \right. \tag{12}$$

From (10), (12) and the relation $\sigma > 1$ it follows that the integral in the right hand member of (11) exists. Further by the calculus of residues we find

$$-\frac{1}{2i} \int_{X-i\infty}^{X+i\infty} (z+w)^{-s} \cot \pi z \, dz = \sum_{X<n\leqq p} \frac{1}{(n+w)^s} - \frac{1}{2i} \int_{K_p} (z+w)^{-z} \cot \pi z \, dz. \tag{13}$$

Again from (10), (12) and the relation $\sigma > 1$ it follows that the last integral in (13) tends to zero for $p \to \infty$. This proves the lemma.

Lemma 2. *Let C be a fixed real number such that*

$$0 < C < 1 \tag{14}$$

and let X and σ_1 be positive. Then, if $s \neq 1$ belongs to the region

$$\sigma \geqq \sigma_1, \ |t| \leqq 2\pi C X,$$

we have for $0 \leqq w \leqq 1$

$$\zeta^*(s, w) = \sum_{1\leqq n<X} \frac{1}{(n+w)^s} - \frac{(X+w)^{1-s}}{1-s} + \Phi, \tag{15}$$

where $|\Phi| < 2\left(1 + \frac{1}{\pi(1-C)}\right) X^{-\sigma}$.

Proof. First we suppose that X is half an odd integer. Then we have by (2) and lemma 1

$$\zeta^*(s, w) = \sum_{1\leqq n<X} \frac{1}{(n+w)^s} - \frac{1}{2i} \int_{X-i\infty}^{X+i\infty} (z+w)^{-s} \cot \pi z \, dz,$$

hence

$$\left\{ \begin{aligned} \zeta^*(s, w) = &\sum_{1\leqq n<X} \frac{1}{(n+w)^s} - \frac{1}{2i} \int_X^{X+i\infty} (z+w)^{-s} (i + \cot \pi z) \, dz \\ &- \frac{1}{2i} \int_{X-i\infty}^{X} (z+w)^{-s} (-i + \cot \pi z) \, dz - \frac{(X+w)^{1-s}}{1-s}. \end{aligned} \right. \tag{16}$$

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[3]) i.e. $(z+w)^{-s} = e^{-s \log(z+w)}$, where $|\text{Im} \log (z+w)| < \pi$.

For $y \geqq 0$, $z = X + iy$, we have

$$0 \leqq \arg(z + w) = \operatorname{arc\,tg} \frac{y}{X+w} \leqq \frac{y}{X+w} \leqq \frac{y}{X},$$

from which, in view of the condition $|t| \leqq 2\pi CX$, it follows

$$(17) \qquad |(z+w)^{-s}| \leqq |z+w|^{-\sigma} e^{\frac{y}{X}|t|} \leqq X^{-\sigma} e^{2\pi C y}.$$

For $y \geqq 0$, $z = X + iy$ we find from (8) and (17)

$$(18) \qquad |(z+w)^{-s}(i + \cot \pi z)| \leqq 2X^{-\sigma} e^{-2(1-C)\pi y}.$$

For $\partial/\partial s\, [(z+w)^{-s}(i + \cot \pi z)]$ a similar estimate holds.

In virtue of these estimates and (14) the first integral in (16) is regular for all s. The same conclusion holds for the second integral in (16). So formula (16) provides the analytic continuation of the function $\zeta^*(s, w)$ over the whole s-plane (except for the point $s = 1$, where there is a single pole with residue 1). In particular (16) holds for $\sigma \geqq \sigma_1$; $s \neq 1$.

Finally, we conclude from (18) and (14),

$$\left| \int_X^{X+i\infty} (z+w)^{-s}(i + \cot \pi z)\, dz \right| \leqq 2X^{-\sigma} \int_0^\infty e^{-2(1-C)\pi y}\, dy = \frac{1}{\pi(1-C)} X^{-\sigma}.$$

As is easily seen, for the second integral in (16) the same estimate holds. Thus in the case that X is half an odd integer we have proved

$$\zeta^*(s, w) = \sum_{1 \leqq n < X} \frac{1}{(n+w)^s} - \frac{(X+w)^{1-s}}{1-s} + \Phi_1,$$

where

$$|\Phi_1| \leqq \frac{2}{\pi(1-C)} X^{-\sigma}.$$

In the general case put $X_1 = X + \vartheta$, where X_1 is half an odd integer and where $0 \leqq \vartheta < 1$. Then the last formula holds with X_1 instead of X. Replacing X_1 by $X = X_1 - \vartheta$, the variation both in the first and in the second term in absolute value is $\leqq X^{-\sigma}$; for the first term this is trivial; for the second term it follows from

$$\left| \frac{(X_1+w)^{1-s}}{1-s} - \frac{(X+w)^{1-s}}{1-s} \right| = \left| \int_{X+w}^{X+\vartheta+w} x^{-s}\, dx \right| \leqq$$

$$\leqq \int_{X+w}^{X+\vartheta+w} x^{-\sigma}\, dx < \vartheta (X+w)^{-\sigma} < X^{-\sigma}.$$

Further we have $X_1^{-\sigma} \leqq X^{-\sigma}$. From this the lemma follows.

Lemma 3. *Let t be real, $|t| \geqq 3$, and let n, m denote positive integers. Put*

$$R_1 = \int_0^1 \Big(\sum_{\substack{n<|t| \\ n \neq m}} \sum_{m<|t|} (n+w)^{-1-it}(m+w)^{-1+it} \Big)\, dw.$$

Then we have

$$|R_1| < 8|t|^{-1} \log |t|.$$

Proof. Without loss of generality we may suppose that t is positive. Let τ be the greatest integer $< t$. Let then R_1^* be defined by

$$(19)\qquad R_1^* = \int_0^1 \Big(\sum_{n<m} \sum_{m<t} (n+w)^{-1-it} (m+w)^{-1+it} \Big)\, dw.$$

We shall prove

$$(20)\qquad |R_1^*| \leqq 4t^{-1} \log t.$$

From this the lemma follows immediately.

In (19) put $m = n + k$. First carrying out the summation over those terms, for which k has a fixed value, we obtain

$$R_1^* = \sum_{k=1}^{\tau-1} \int_0^1 \Big(\sum_{n=1}^{\tau-k} (n+w)^{-1-it} (n+k+w)^{-1+it} \Big)\, dw$$
$$= \sum_{k=1}^{\tau-1} \int_1^{\tau-k+1} v^{-1-it} (v+k)^{-1+it}\, dv$$
$$= \sum_{k=1}^{\tau-1} \int_1^{\tau-k+1} e^{-it \log v/k+v} \cdot \frac{1}{v(k+v)}\, dv.$$

Now $u = \log v/k + v$ is a monotoneously increasing function of v which has a derivative $k/v(k+v)$. Hence it follows

$$R_1^* = \sum_{k=1}^{\tau-1} \frac{1}{k} \int_{\log 1/k+1}^{\log(1-k/\tau+1)} e^{-itu}\, du = \sum_{k=1}^{\tau-1} \frac{2\theta_k}{kt},$$

where $|\Theta_k| \leqq 1$.

From this (20) follows; so the lemma is proved.

Lemma 4. *Let σ, t be real, and let n, m denote positive integers. Put*

$$(21)\qquad R_\sigma = \int_0^1 \Big(\sum_{\substack{n<t \\ n \neq m}} \sum_{m<|t|} (n+w)^{-\sigma-it} (m+w)^{-\sigma+it} \Big)\, dw.$$

Then, if

$$\tfrac{1}{2} \leqq \sigma \leqq 1,\ |t| \geqq 3,$$

we have

$$|R_\sigma| \leqq 20\, |t|^{1-2\sigma} \log |t|.$$

Proof. Again it is no loss of generality to take t positive. Now we define

$$R_\sigma^* = \int_0^1 \Big(\sum_{n<m} \sum_{m<t} (n+w)^{-\sigma-it} (m+w)^{-\sigma+it} \Big)\, dw.$$

Introducing

$$f_k(v) = v^{1-\sigma} (k+v)^{1-\sigma},\ \varphi_k(v) = \frac{1}{v(k+v)}\, e^{-it \log v/k+v},$$

we find

$$R_\sigma^* = \sum_{k=1}^{\tau-1} \int_1^{\tau-k+1} v^{-\sigma-it} (k+v)^{-\sigma+it}\, dv$$
$$= \sum_{k=1}^{\tau-1} \int_1^{\tau-k+1} f_k(v)\, \varphi_k(v)\, dv$$
$$= \sum_{k=1}^{\tau-1} Q(k), \text{ say.}$$

Here $f_k(v)$ is a positive, monotoneously increasing function and $\varphi_k(v)$ is a complex function which for all a with $0 < a < \tau - k + 1$ satisfies the relation

$$\int_a^{\tau-k+1} \varphi_k(v)\, dv = \frac{2\Theta_k}{kt}, \text{ where } |\Theta_k| \leqq 1.$$

By BONNET's mean-value theorem we have

$$Q_k = \int_1^{\tau-k+1} f_k(v)\cdot Re\, \varphi_k(v)\, dv + i \int_1^{\tau-k+1} f_k(v)\cdot Im\, \varphi_k(v)\, dv$$
$$= f(\tau - k + 1)\cdot[Re \int_{\xi_1}^{\tau-k+1} \varphi_k(v)\, dv + i\, Im \int_{\xi_2}^{\tau-k+1} \varphi_k(v)\, dv]$$

for some ξ_1, ξ_2 with $0 < \xi_j < \tau - k + 1$ $(j = 1, 2)$.

Hence it follows

$$|Q(k)| \leqq (\tau - k + 1)^{1-\sigma}(\tau + 1)^{1-\sigma}\cdot\frac{4}{kt} < \frac{4}{k}\, t^{-\sigma}(t+1)^{1-\sigma} < \frac{5}{k}\, t^{1-2\sigma},$$

in view of the conditions of the lemma. Hence we have

$$|R_\sigma^*| \leqq 10\, t^{1-2\sigma} \log t.$$

From this the lemma follows immediately.

Lemma 5. *If τ is a positive integer and $\frac{1}{2} \leqq \sigma \leqq 1$, then*

$$\int_0^1 \Big(\sum_{n=1}^{\tau} (n+w)^{-\sigma}\Big)\, dw \leqq (\tau+1)^{1-\sigma} \log(\tau+1). \tag{22}$$

Proof. If $\frac{1}{2} \leqq \sigma < 1$, we have

$$\int_0^1 \Big(\sum_{n=1}^{\tau} (n+w)^{-\sigma}\Big)\, dw = \sum_{n=1}^{\tau} \Big(\frac{(n+w)^{1-\sigma}}{1-\sigma}\Big|_0^1\Big)$$
$$= \frac{1}{1-\sigma}\{(\tau+1)^{1-\sigma} - 1\} = (\tau+1)^{1-\sigma}\log(\tau+1)\cdot\frac{1-(\tau+1)^{-(1-\sigma)}}{(1-\sigma)\log(\tau+1)}.$$

Here $u = (1-\sigma)\log(\tau+1)$ is positive. Thus we have $e^u > 1 + u$, hence

$$\frac{d}{du}\,\frac{1-e^{-u}}{u} = \frac{(u+1)\,e^{-u}-1}{u^2} < 0 \text{ if } u > 0.$$

Further $\frac{1-e^{-u}}{u} \to 1$ for $u \to 0$, hence

$$\frac{1-(\tau+1)^{-(1-\sigma)}}{(1-\sigma)\log(\tau+1)} = \frac{1-e^{-u}}{u} < 1.$$

This proves (22) in the case $\frac{1}{2} \leqq \sigma < 1$. If $\sigma = 1$, clearly (22) is valid with the equality sign. So the lemma is proved.

Proof of theorems 1 and 2. We assume $\frac{1}{2} \leqq \sigma \leqq 1$ and $|t| \geqq 3$. Let n, m denote positive integers. Applying lemma 2 with $X = |t|$, $C = (2\pi)^{-1}$ we infer

$$\zeta^*(s, w) = \sum_{n<|t|} \frac{1}{(n+w)^s} + \Phi^*,$$

where

(23) $$|\Theta^*| \leqq \left\{2 + \frac{4}{2\pi-1}\right\} |t|^{-\sigma} < 3\,|t|^{-\sigma}.$$

Hence we have

(24) $$\left\{\begin{aligned} &\int_0 |\zeta^*(s, w)|^2\, dw = \int_0^1 |\sum_{n<|t|} (n+w)^{-s} + \Phi^*|^2\, dw \\ &= \int_0^1 (\sum_{n<|t|} (n+w)^{-2\sigma})\, dw + \int_0^1 (\sum_{\substack{n<|t|\\ n\neq m}} \sum_{m<|t|} (n+w)^{-\sigma-it}(m+w)^{-\sigma+it})\, dw \\ &+ 2\, Re \int_0^1 (\Phi^* \cdot \sum_{n<|t|} (n+w)^{-\sigma+it})\, dw + \int_0^1 |\Phi^*|^2\, dw \\ &= T_1 + T_2 + T_3 + T_4, \text{ say.}\end{aligned}\right.$$

By lemma 4 we have $|T_2| \leqq 20\,|t|^{1-2\sigma} \log\,|t|$. Further it follows from

(23) $$|T_4| = T_4 < 9\,|t|^{-2\sigma} < 3\,|t|^{1-2\sigma} \log\,|t|.$$

For T_3 we find, in virtue of (23), lemma 5 and the condition $|t| \geqq 3$

$$|T_3| \leqq 6\,|t|^{-\sigma} \int_0^1 \sum_{n<|t|} (n+w)^{-\sigma}\, dw <$$
$$< 6\,|t|^{-\sigma} (|t|+1)^{1-\sigma} \log(|t|+1) < 9\,|t|^{1-2\sigma} \log|t|.$$

In estimating T_1 we distinguish two cases:

a) $\frac{1}{2} + \frac{1}{2A\log|t|} \leqq \sigma \leqq 1$, hence $\frac{1}{2\sigma-1} \leqq A \log|t|$.

Then we have

$$\int_0^1 \sum_{n<|t|} (n+w)^{-2\sigma}\, dw = \sum_{n<|t|} \frac{(n+w)^{1-2\sigma}}{1-2\sigma}\Big|_0^1$$
$$= \frac{1}{2\sigma-1} - \frac{(\tau+1)^{1-2\sigma}}{2\sigma-1} = \frac{1}{2\sigma-1} + \Theta A\,|t|^{1-2\sigma} \log|t|,$$

where $|\Theta| \leqq 1$. Summing up the results (24) yields

$$\int_0^1 |\zeta^*(s, w)|^2\, dw = \frac{1}{2\sigma-1} + \Theta\,(A+32)\,|t|^{1-2\sigma} \log|t|,$$

where $|\Theta| \leqq 1$. Theorem 1 now follows at once.

b) $\sigma = \frac{1}{2}$. Then we have

$$\int_0^1 \sum_{n<|t|} (n+w)^{-2\sigma}\, dw = \log(\tau+1) < 2\log|t|.$$

Summarizing we find

$$\int_0^1 |\zeta^*(\tfrac{1}{2}+it, w)|^2\, dw < 34 \log\,|t|.$$

This proves theorem 2.

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